Journal of Statistical Physics, Vol. 100, Nos. 3/4, 2000

New Exact Ground States for One-Dimensional Quantum Many-Body Systems

Thomas Koprucki[1] **and Heinz-Jürgen Wagner**[2]

Received January 20, 2000

We consider one-dimensional quantum many-body systems with pair interactions in external fields and (re)investigate the conditions under which exact ground-state wave functions of product type can be found. Contrary to a claim in the literature that an exhaustive list of such systems is already known, we show that this list can still be enlarged considerably. In particular, we are able to calculate exact ground-state wave functions for a class of quantum many-body systems with $Ax^{-2}+Bx^2$ interaction potentials and external potentials given by sixth-order polynomials.

KEY WORDS: Ground state; wave functions of product type; Calogero–Sutherland systems.

1. INTRODUCTION

A large amount of attention has been devoted to the properties of several types of exactly solvable interacting one-dimensional quantum many-body systems. This is mostly due to the fact that such systems and their classical counterparts show up in a large number of physical problems which seem to be rather disparate at first glance. To point out only a few of them, we mention the connection to random matrix theory,[1] the description of one-dimensional Wigner crystals,[2, 3] and the theory of Heisenberg spin chains.[4, 5] For a more extensive list of such problems—covering topics of field theory as well—see, e.g., the introduction of ref. 6 and the literature given there.

In this paper we are concerned with the particular problem of finding and classifying interacting one-dimensional quantum many-body systems

[1] Weierstraß-Institut für Angewandte Analysis und Stochastik, 10117 Berlin, Germany.
[2] Fachbereich 6-Theoretische Physik, Universität Paderborn, 33098 Paderborn, Germany.

0022-4715/00/0800-0779$18.00/0

with exact ground-state wave functions of product type. Sutherland[7] was the first to point out that a ground-state wave function of the form

$$\Psi(x_1,..., x_N) = \prod_{1 \leqslant i < j \leqslant N} \chi(x_i - x_j); \qquad \chi(-x) = \pm\chi(x) \tag{1.1}$$

is an eigenfunction of a many-particle Hamiltonian with pair interactions only, i.e.,

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{\substack{i,\, j=1 \\ i \neq j}}^{N} W(x_i - x_j) \tag{1.2}$$

whenever the (necessarily odd) logarithmic derivative

$$\varphi(x) = \chi'(x)/\chi(x) \tag{1.3}$$

satisfies the functional equation

$$\begin{aligned} &\varphi(x)\, \varphi(y) + \varphi(y)\, \varphi(z) + \varphi(z)\, \varphi(x) \\ &\quad = f(x) + f(y) + f(z) \qquad \text{for} \quad x + y + z = 0 \end{aligned} \tag{1.4}$$

The interaction potential W is then given by

$$W(x) = \frac{\hbar^2}{m} (\varphi'(x) + \varphi^2(x) - (N-2)\, f(x)) + \text{const.} \tag{1.5}$$

The general meromorphic solution of the above functional equation has been found by Calogero[8] and reads

$$\varphi(x) = \alpha\zeta(x;\, g_2,\, g_3) + \beta x; \tag{1.6}$$

$$f(x) = -\frac{1}{2} \left\{ \alpha^2 \frac{d\zeta}{dx} (x;\, g_2,\, g_3) + \alpha^2 \zeta^2(x;\, g_2,\, g_3) + \beta^2 x^2 + 2\alpha\beta x \zeta(x;\, g_2,\, g_3) \right\} \tag{1.7}$$

Here, $\zeta(x;\, g_2,\, g_3)$ denotes the Weierstraß zeta function[9] with power series expansion

$$\zeta(x;\, g_2,\, g_3) = \frac{1}{x} - \frac{g_2}{2^2 \cdot 3 \cdot 5}\, x^3 - \frac{g_3}{2^2 \cdot 5 \cdot 7}\, x^5 + \mathcal{O}(x^7) \tag{1.8}$$

On a more general level one can then deal with the question whether there are systems with ground-state wave functions of the type

$$\Psi(x_1,..., x_N) = \prod_{i=1}^{N} \sigma(x_i) \prod_{1 \leqslant i < j \leqslant N} \chi(x_i - x_j); \qquad \chi(-x) = \pm\chi(x) \tag{1.9}$$

that are exact eigenfunctions of Hamiltonians of the form

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} V(x_i) + \frac{1}{2} \sum_{\substack{i, j=1 \\ i \neq j}}^{N} W(x_i - x_j) \tag{1.10}$$

Generalizing Calogero's considerations, Inozemtsev and Meshcheryakov[10] were able to make up a list of such systems and even claimed it to be exhaustive. However, as Forrester recently pointed out,[3] this last statement cannot be correct. The Hamiltonian and the corresponding exact ground-state wave function employed in ref. 3 for the description of a one-dimensional Wigner solid have the form given in equations (1.10) and (1.9) but nevertheless do not show up in the above-mentioned list!

This observation most obviously shows the necessity of a reexamination of the arguments given by Inozemtsev and Meshcheryakov. This will be one of the topics of the present paper. In Section 2 it is pointed out where the reasoning in ref. 10 turns out to be too restrictive and how it can be generalized. In the following sections we then show how to construct a whole class of new Hamiltonians of the form (1.10) with exact eigenfunctions of type (1.9). In particular, we demonstrate that it is possible to calculate exact ground-state wave functions for several quantum many-body systems with interaction potentials $W(x) = Ax^{-2} + Bx^2$ and external potentials $V(x)$ that are given by sixth-order polynomials. Furthermore, it will turn out that Forrester's example of ref. 3 is also covered by our new class of Hamiltonians.

2. THE INOZEMTSEV–MESHCHERYAKOV FUNCTIONAL EQUATION AND ITS GENERALIZATION

We do not intend to repeat the considerations of ref. 10 in every detail here, but restrict ourselves to the necessary minimum of steps. Inserting H from (1.10), Ψ from (1.9), and using the obvious identity

$$\frac{1}{\Psi} \frac{\partial^2 \Psi}{\partial x_k^2} = \frac{\partial}{\partial x_k} \left(\frac{1}{\Psi} \frac{\partial \Psi}{\partial x_k} \right) + \left(\frac{1}{\Psi} \frac{\partial \Psi}{\partial x_k} \right)^2 \tag{2.1}$$

we get

$$H\Psi = \sum_i \left[V(x_i) - \frac{\hbar^2}{2m}\,\tau'(x_i) - \frac{\hbar^2}{2m}\,\tau^2(x_i) \right] \Psi$$
$$+ \frac{1}{2} \sum_{i \neq j} \left[W(x_i - x_j) - \frac{\hbar^2}{m}\,\varphi'(x_i - x_j) \right] \Psi - \frac{\hbar^2}{2m} \sum_i \Bigg(\sum_{\substack{j \\ j \neq i}} \varphi(x_i - x_j) \Bigg)^2 \Psi$$
$$- \frac{\hbar^2}{2m} \sum_{i \neq j} \frac{1}{2}\,\varphi(x_i - x_j)(\tau(x_i) - \tau(x_j))\,\Psi \tag{2.2}$$

where we introduced the logarithmic derivatives

$$\varphi(x) = \chi'(x)/\chi(x), \qquad \tau(x) = \sigma'(x)/\sigma(x) \tag{2.3}$$

We are interested in wave functions Ψ for which (2.2) is reducible to the eigenvalue equation $H\Psi = E\Psi$ by a proper choice of V and W. This can be achieved, if the third and the fourth term on the r.h.s. of (2.2) can be rewritten as sums of one- and/or two-particle potentials. The third term is well-known from the case of vanishing external potential V. It reduces to a sum of two-particle potentials, if we assume the functional equation (1.4) to hold:

$$\sum_i \Bigg(\sum_{\substack{j \\ j \neq i}} \varphi(x_i - x_j) \Bigg)^2 = \sum_{i \neq j} (\varphi^2(x_i - x_j) - (N-2)\, f(x_i - x_j)) \tag{2.4}$$

Inozemtsev and Meshcheryakov[10] then in addition required the validity of the functional equation

$$\varphi(x - y)(\tau(x) - \tau(y)) = \lambda(x) + \lambda(y) \tag{2.5}$$

and managed to determine its solutions.

The introduction of (2.5) indeed leads to solutions of the original problem, as the fourth term on the r.h.s. of (2.2) then reduces to a sum of one-particle potentials. However, at this point we can already demonstrate why the class of solutions in ref. 10 turns out to be too restrictive. The requirement of (2.5) is by no means necessary! A reduction to a sum of two-particle potentials via

$$\varphi(x - y)(\tau(x) - \tau(y)) = F(x - y); \qquad F(x) = F(-x) \tag{2.6}$$

would also do the job. Even more general is the case where the last term on the r.h.s. of (2.2) can be written as a sum of one-particle plus a sum of two-particle potentials. This can be achieved by employing the functional equation

$$\varphi(x-y)(\tau(x)-\tau(y))=\lambda(x)+\lambda(y)+F(x-y); \qquad F(x)=F(-x) \tag{2.7}$$

instead of (2.5) or (2.6), since

$$\sum_{i\neq j} \tfrac{1}{2}\varphi(x_i-x_j)(\tau(x_i)-\tau(x_j))=(N-1)\sum_i \lambda(x_i)+\tfrac{1}{2}\sum_{i\neq j} F(x_i-x_j) \tag{2.8}$$

in this case. Thus, given the validity of (1.4) and (2.7), Eq. (2.2) can be cast into the form

$$\begin{aligned} H\Psi=&\sum_i \left[V(x_i)-\frac{\hbar^2}{2m}(\tau'(x_i)+\tau^2(x_i)+(N-1)\,\lambda(x_i))\right]\Psi \\ &+\frac{1}{2}\sum_{i\neq j}\left[W(x_i-x_j)-\frac{\hbar^2}{m}\varphi'(x_i-x_j)-\frac{\hbar^2}{m}\varphi^2(x_i-x_j)\right. \\ &\left.+\frac{\hbar^2}{m}(N-2)\,f(x_i-x_j)-\frac{\hbar^2}{2m}F(x_i-x_j)\right]\Psi \end{aligned} \tag{2.9}$$

Here one can immediately read off the choices that have to be made for V and W in order to achieve the desired equation $H\Psi=E\Psi$.

The results obtained so far can thus be summed up as follows:

Proposition 1. Given a Hamiltonian H of the form (1.10) and a wave function Ψ of type (1.9), the following requirements are sufficient for the eigenvalue equation $H\Psi=E\Psi$ to hold:

(a) The logarithmic derivatives $\varphi=\chi'/\chi$ and $\tau=\sigma'/\sigma$ are solutions of the functional equations

$$\begin{aligned} &\varphi(x)\,\varphi(y)+\varphi(y)\,\varphi(z)+\varphi(z)\,\varphi(x) \\ &\quad=f(x)+f(y)+f(z) \qquad \text{for} \quad x+y+z=0 \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\varphi(x-y)(\tau(x)-\tau(y)) \\ &\quad=\lambda(x)+\lambda(y)+F(x-y), \qquad F(x)=F(-x) \end{aligned} \tag{2.11}$$

(b) The one- and two-particle potentials V and W are given by

$$V(x)=\frac{\hbar^2}{2m}(\tau'(x)+\tau^2(x)+(N-1)\lambda(x))+\frac{E_1}{N} \tag{2.12}$$

$$W(x)=\frac{\hbar^2}{m}\left(\varphi'(x)+\varphi^2(x)-(N-2)\,f(x)+\frac{1}{2}F(x)\right)+\frac{2(E-E_1)}{N(N-1)} \tag{2.13}$$

3. SOLUTION OF THE GENERALIZED FUNCTIONAL EQUATION

In this section we deal with the question how the solutions of the functional equation system (2.10)–(2.11) can be found. As was already mentioned in the introduction, the general meromorphic solution of (2.10) is given by[8]

$$\varphi(x) = \alpha\zeta(x;\, g_2,\, g_3) + \beta x \tag{3.1}$$

To solve (2.11) we now apply the methods used in refs. 8 and 10 for the solutions of the functional equations (1.4) and (2.5). In particular, we shall derive ordinary differential equations for τ or λ alone which are among a couple of necessary conditions for the functional equation (2.11) to hold. In a next step, a class of possible solutions $\varphi(x)$ is determined. In some cases it will turn out to be only a subclass of the functions given by (3.1) due to restrictions which have to be imposed upon the parameters g_2, g_3. Subsequently, the general solution of the differential equation for τ is given. Finally it is pointed out that the functions $\varphi(x)$ and $\tau(x)$ traced out by the above strategy are already solutions of (2.11).

To arrive at physically meaningful expressions, we require τ and λ both to be nonsingular and sufficiently smooth. Furthermore, it turns out to be advantageous to investigate the cases $\lambda = \text{const.}$ and $\lambda \neq \text{const.}$ separately.

3.1. The Case $\lambda = \lambda_0 = \text{const}$

For constant $\lambda = \lambda_0$, (2.11) reduces to

$$\varphi(x-y)(\tau(x) - \tau(y)) = 2\lambda_0 + F(x-y) \tag{3.2}$$

Making the special choice $y = x + \varepsilon$, we get

$$\varphi(\varepsilon)(\tau(x+\varepsilon) - \tau(x)) = 2\lambda_0 + F(\varepsilon) \tag{3.3}$$

Differentiation with respect to x then leads to $\tau'(x+\varepsilon) - \tau'(x) = 0$, or

$$\tau'(x) = \text{const.} \tag{3.4}$$

Inserting the general solution $\tau(x) = \tau_1 + \tau_2 x$ into the original functional equation (2.11), we see that it is satisfied with $\lambda = \lambda_0$ and $F(x) = \tau_2 x\varphi(x) - 2\lambda_0$. Thus we have shown:

Proposition 2. The functional equation system (2.10)–(2.11) is solved by

$$\varphi(x) = \alpha\zeta(x;\, g_2,\, g_3) + \beta x; \qquad \tau(x) = \tau_1 + \tau_2 x \tag{3.5}$$

with

$$\lambda = \lambda_0 = \text{const.}; \qquad F(x) = \tau_2 x(\alpha\zeta(x;\, g_2,\, g_3) + \beta x) - 2\lambda_0 \tag{3.6}$$

3.2. The Case $\lambda = \lambda(x) \neq$ const

For $\lambda(x) \neq$ const. we again start with writing down (2.11) for the special choice $y = x + \varepsilon$ and with φ given by (3.1):

$$\varphi(\varepsilon)(\tau(x+\varepsilon) - \tau(x)) = \lambda(x+\varepsilon) + \lambda(x) + F(\varepsilon) \tag{3.7}$$

Both sides of this expression are now expanded into a power series in ε up to order ε^6 and subsequently a comparison of coefficients is carried through.

Since F is even, we can write

$$F(\varepsilon) = F_0 + F_2\varepsilon^2 + F_4\varepsilon^4 + F_6\varepsilon^6 + \mathcal{O}(\varepsilon^8) \tag{3.8}$$

Furthermore, from (3.1) and (1.8), one has

$$\varphi(\varepsilon) = \alpha\left(\frac{1}{\varepsilon} - \frac{g_2}{2^2 \cdot 3 \cdot 5}\,\varepsilon^3 - \frac{g_3}{2^2 \cdot 5 \cdot 7}\,\varepsilon^5\right) + \beta\varepsilon + \mathcal{O}(\varepsilon^7) \tag{3.9}$$

In addition we have of course to employ the Taylor expansions of τ up to seventh and of λ up to sixth order.

In zeroth order the comparison of coefficients leads to

$$\alpha\tau'(x) = 2\lambda(x) + F_0 \tag{3.10}$$

Since $\lambda(x) \neq$ const., this implies $\alpha \neq 0$. (3.10) can therefore be used in the following as a tool for replacing derivatives $\tau^{(n)}(x)$ with derivatives $\lambda^{(n-1)}(x)$ and vice versa.

Comparison of the first order coefficients yields

$$\frac{\alpha}{2}\,\tau''(x) = \lambda'(x) \tag{3.11}$$

This is automatically satisfied whenever (3.10) is valid.

In second order we are led to

$$\frac{\alpha}{6}\,\tau'''(x) + \beta\tau'(x) = \frac{1}{2}\,\lambda''(x) + F_2 \tag{3.12}$$

Using (3.10), we thus get

$$\lambda''(x) - 12\frac{\beta}{\alpha}\lambda(x) = 6\left(\frac{\beta}{\alpha}F_0 - F_2\right) \tag{3.13}$$

and

$$\tau'''(x) - 12\frac{\beta}{\alpha}\tau'(x) = -\frac{12}{\alpha}F_2 \tag{3.14}$$

These are the already proclaimed differential equations for λ and τ alone.

The third order again leads to nothing new, as insertion of (3.10) implies

$$\lambda'''(x) - 12\frac{\beta}{\alpha}\lambda'(x) = 0 \tag{3.15}$$

which is satisfied due to (3.13).

After some manipulations, the fourth order gives rise to the equation

$$\left[12\left(\frac{\beta}{\alpha}\right)^2 - g_2\right]\lambda(x) = \frac{1}{2}F_0\left(g_2 - 12\left(\frac{\beta}{\alpha}\right)^2\right) + 30F_4 + 6\frac{\beta}{\alpha}F_2 \tag{3.16}$$

Since λ is nonconstant, this equation can only be satisfied, if

$$g_2 = 12\left(\frac{\beta}{\alpha}\right)^2 \tag{3.17}$$

and subsequently leads to a relation between F_4 and F_2:

$$F_4 = -\frac{\beta}{5\alpha}F_2 \tag{3.18}$$

By employing the above results, it can be shown that the equation corresponding to the fifth order is again automatically satisfied.

The sixth order yields

$$\begin{aligned}&\left[\left(12\frac{\beta^2}{\alpha^2} - \frac{7}{6}g_2\right)12\frac{\beta}{\alpha} - 3g_3\right]\lambda(x)\\ &\quad = \frac{3}{2}g_3F_0 + 210F_6 - 6\left(F_0\frac{\beta}{\alpha} - F_2\right)\left(12\frac{\beta^2}{\alpha^2} - \frac{7}{6}g_2\right)\end{aligned} \tag{3.19}$$

With $\lambda \neq$ const. and after elimination of g_2 with the aid of (3.17), this leads to

$$g_3 = -8\left(\frac{\beta}{\alpha}\right)^3 \tag{3.20}$$

and

$$F_6 = -\frac{1}{105}\left(\frac{\beta}{\alpha}\right)^2 F_2 \tag{3.21}$$

As the parameters g_2, g_3 are already fixed by (3.17) and (3.20), the class of functions $\varphi(x)$ is now reduced to

$$\varphi(x) = \alpha\zeta\left(x;\, 12\left(\frac{\beta}{\alpha}\right)^2,\, -8\left(\frac{\beta}{\alpha}\right)^3\right) + \beta x \tag{3.22}$$

A degenerate case of the Weierstraß zeta function shows up here,[9] and we can rewrite (3.22) as follows:

$$\varphi(x) = \begin{cases} \alpha\gamma\cot(\gamma x) & \text{for } \dfrac{\beta}{\alpha} < 0 \\ \dfrac{\alpha}{x} & \text{for } \dfrac{\beta}{\alpha} = 0, \quad \gamma = \sqrt{3\left|\dfrac{\beta}{\alpha}\right|} \\ \alpha\gamma\coth(\gamma x) & \text{for } \dfrac{\beta}{\alpha} > 0 \end{cases} \tag{3.23}$$

The class of functions $\tau(x)$ is limited by (3.14). The general solution of this differential equation is given by

$$\tau(x) = \begin{cases} \tau_1\cos(2\gamma x) + \tau_2\sin(2\gamma x) + \tau_3 + \tau_4 x & \text{for } \dfrac{\beta}{\alpha} < 0 \\ \tau_1 + \tau_2 x + \tau_3 x^2 + \tau_4 x^3 & \text{for } \dfrac{\beta}{\alpha} = 0, \ \gamma = \sqrt{3\left|\dfrac{\beta}{\alpha}\right|} \\ \tau_1\cosh(2\gamma x) + \tau_2\sinh(2\gamma x) + \tau_3 + \tau_4 x & \text{for } \dfrac{\beta}{\alpha} > 0 \end{cases} \tag{3.24}$$

Note that there is the following connection between F_2 and τ_4:

$$F_2 = \beta\tau_4 \quad \text{for } \frac{\beta}{\alpha} \neq 0; \qquad F_2 = -\frac{\alpha}{2}\tau_4 \quad \text{for } \frac{\beta}{\alpha} = 0 \tag{3.25}$$

If one now inserts the expressions (3.23) and (3.24) for φ and τ into the functional equation (2.11), one sees that it is already fulfilled, that is, we have

Proposition 3. The functional equation system (2.10)–(2.11) is solved by $\varphi(x)$ given by (3.23) and $\tau(x)$ given by (3.24).

As we do not want to overburden the paper, we refrain from giving the corresponding expressions for $\lambda(x)$ and $F(x)$ explicitly.

Remark. The solutions of the functional equation (2.5) can be recovered from the results of this subsection by putting $\tau_4 = 0$.

4. DISCUSSION

The results of the previous section lead to a couple of new Hamiltonians H of the form (1.10) and wave functions Ψ of product type (1.9) which obey the eigenvalue equation $H\Psi = E\Psi$. However, not all of these results are physically meaningful, as it sometimes may happen that we are led to functions Ψ that are not square-integrable and thus cannot be interpreted as eigenfunctions. This problem has always to be discussed for the concrete particular case under study.

To carry through such a discussion at least for one of the most interesting special cases, let us consider the functions

$$\varphi(x) = \frac{\alpha}{x} \tag{4.1}$$

$$\tau(x) = \tau_1 + \tau_2 x + \tau_3 x^2 + \tau_4 x^3 \tag{4.2}$$

This corresponds to the choice $\beta/\alpha = 0$ in (3.23) and (3.24).

Inserting (4.1) and (4.2) into the l.h.s. of the functional equation (2.11), we get

$$\begin{aligned}(\tau(x) - \tau(y))\varphi(x-y) &= \alpha\,\frac{\tau_2(x-y) + \tau_3(x^2-y^2) + \tau_4(x^3-y^3)}{x-y} \\ &= \alpha\left(\tau_2 + \tau_3(x+y) + \tau_4\left(\frac{3}{2}x^2 + \frac{3}{2}y^2 - \frac{1}{2}(x-y)^2\right)\right)\end{aligned} \tag{4.3}$$

It is immediately read off that λ and F can be chosen as

$$\lambda(x) = \alpha \left(\frac{1}{2} \tau_2 + \tau_3 x + \frac{3}{2} \tau_4 x^2 \right) \tag{4.4}$$

$$F(x) = -\frac{\alpha}{2} \tau_4 x^2 \tag{4.5}$$

Inserting (4.2) and (4.4) into (2.12), one immediately recognizes that the external potential $V(x)$ is given by a certain sixth order polynomial. Proper choice of the parameters leads to a whole bunch of interesting—symmetric as well as nonsymmetric—double and triple well potentials here.

Furthermore, from (2.10) one can find out that $f(x) = 0$ for $\varphi(x) = \alpha/x$. Together with (4.1), (4.5) this leads via (2.13) to the following interaction potential:

$$W(x) = \frac{\hbar^2}{m} \left\{ \frac{\alpha(\alpha-1)}{x^2} - \frac{\alpha\tau_4}{4} x^2 \right\} + \text{const.} \tag{4.6}$$

Integrating the logarithmic derivatives $\tau = \sigma'/\sigma$ and $\varphi = \chi'/\chi$, we finally arrive at the following expressions for the factors from which the wave function Ψ is built up:

$$\chi(x) = C_1 \, |x|^\alpha; \qquad \sigma(x) = C_2 \exp\left(\tau_1 x + \frac{\tau_2}{2} x^2 + \frac{\tau_3}{3} x^3 + \frac{\tau_4}{4} x^4 \right); \tag{4.7}$$
$$C_1, C_2 = \text{const.}$$

Square integrability of Ψ at infinity can be ensured by putting $\tau_4 < 0$. Moreover, as Ψ turns out to be nodeless outside the hyperplanes $x_i - x_j = 0$, it can be regarded as ground state.[11, 12]

Remark. For $\tau_4 = 0$ we are not furnished with anything new, as square integrability of Ψ can only be achieved in this case, if we require $\tau_3 = 0$, $\tau_2 < 0$. The external potential V is then reduced to a harmonic well, and the corresponding system is already well-known.[12]

We now conclude this paper by pointing out in which of the above new classes of solutions one can find Forrester's (counter-)example from ref. 3. To this end we start with the solutions φ and τ from Proposition 2 in Subsection 3.1 above, i.e.,

$$\varphi(x) = \alpha\zeta(x;\, g_2, g_3) + \beta x\,; \qquad \tau(x) = \tau_1 + \tau_2 x \tag{4.8}$$

Putting $\tau_1 = 0$ and making the transition to the degenerate case where

$$g_2 = 12\left(\frac{\beta}{\alpha}\right)^2, \qquad g_3 = -8\left(\frac{\beta}{\alpha}\right)^3, \qquad \frac{\beta}{\alpha} > 0 \tag{4.9}$$

we are led to (cf. (3.22), (3.23))

$$\varphi(x) = \alpha \sqrt{3\frac{\beta}{\alpha}} \coth\left(\sqrt{3\frac{\beta}{\alpha}}\, x\right), \qquad \tau(x) = \tau_2 x \tag{4.10}$$

Choosing $\tau_2 < 0$ for integrability reasons, we exactly end up with the type of solutions discussed by Forrester.

REFERENCES

1. T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Random-matrix theories in quantum physics: Common concepts, *Phys. Rep.* **299**:190 (1998).
2. B. Sutherland, One-dimensional plasma as an example of a Wigner solid, *Phys. Rev. Lett.* **35**:185 (1975).
3. P. J. Forrester, Properties of an exact crystalline many-body ground state, *J. Statist. Phys.* **76**:331 (1994).
4. F. D. M. Haldane, Exact Jastrow–Gutzwiller resonating-valence-bond ground state of the spin-$\frac{1}{2}$ antiferromagnetic Heisenberg chain with $1/r^2$ exchange, *Phys. Rev. Lett.* **60**:635 (1988).
5. B. S. Shastry, Exact solution of an $S = \frac{1}{2}$ Heisenberg antiferromagnetic chain with long-ranged interactions, *Phys. Rev. Lett.* **60**:639 (1988).
6. C. Quesne, Knizhnik–Zamolodchikov equations and the Calogero–Sutherland–Moser integrable models with exchange terms, *J. Phys. A: Math. Gen.* **28**:3533 (1995).
7. B. Sutherland, Exact results for a quantum many-body problem in one dimension, *Phys. Rev. A* **4**:2019 (1971).
8. F. Calogero, One-dimensional many-body problems with pair interactions whose exact ground-state wave function is of product type, *Lett. Nuovo Cimento* **13**:507 (1975).
9. A. Erdélyi (ed.), *Higher Transcendental Functions*, Vol. II, Sections 13.12–13.15 (McGraw–Hill, 1953).
10. V. I. Inozemtsev and D. V. Meshcheryakov, On ground-state wavefunctions for Sutherland–Calogero systems in an external field, *Phys. Lett. A* **106**:101 (1984).
11. F. Calogero, Ground state of a one-dimensional N-body system, *J. Math. Phys.* **10**: 2197 (1969).
12. B. Sutherland, Quantum many-body problem in one dimension: Ground state, *J. Math. Phys.* **12**:246 (1971).